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Fourier-Mukai transforms and canonical divisors

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1 Introduction

Let X be a smooth projective variety over $\mathbf{C}$. The main subject of this article is investigating the derived category of coherent sheaves on X. We define $D(X)$ as

$$D(X) := D^b(\mathrm{Coh}(X)).$$

Recently $D(X)$ has attained much interest in many mathematical aspects, and is expected to represent several symmetries. We have the following examples:

- Homological mirror symmetry

  If X is a Calabi-Yau 3-fold, it is expected there exists another Calabi-Yau 3-fold $\check{X}$, such that X and $\check{X}$ are related by some symmetry. $\check{X}$ is called a mirror manifold of X. It is expected we can define Fukaya category $Fuk(\check{X})$, which depends not complex structures but symplectic structures, such that we have an equivalence
  $$D(X) \to DFuk(\check{X}).$$

- Moduli spaces of stable sheaves

  $D(X)$ is useful in investigating moduli spaces of stable sheaves. It is well-known that some moduli spaces have equivalent derived categories with that of the original varieties. The famous example of Mukai [8] shows if A is an abelian variety and $\hat{A}$ is its dual, the Poincare line bundle gives an equivalence
  $$D(\hat{A}) \to D(A).$$

- Birational geometry

  Let $\phi \colon X \dashrightarrow X^+$ be a 3-dimensional flop. In [2], Bridgeland showed there exists an equivalence
  $$D(X) \to D(X^+).$$
  His method is considering X^+ as a moduli space of perverse sheaves $\mathrm{Per}(X) \subset D(X)$. It is remarkable that his method gives a conceptual proof of the existence of flops.

In this article, we are concerned with the problem "To what extent is X determined by $D(X)$?" We define $FM(X)$ as the set of smooth projective varieties, whose derived categories are equivalent to $D(X)$, up to isomorphism. The members of $FM(X)$ are called Fourier-Mukai partners of X. Take an object $\mathcal{P} \in D(X \times Y)$. Then we can define a functor

$$\Phi^{\mathcal{P}}_{X \to Y} := \mathbf{R}g_*(f^*(*) \overset{\mathbf{L}}{\otimes} \mathcal{P}) \colon D(X) \to D(Y).$$

Here $f\colon X \times Y \to X$ and $g\colon X \times Y \to Y$ are projections. The following theorem is fundamental.

Theorem 1.1 (Orlov [11]) *Let $Y \in FM(X)$ and $\Phi\colon D(X) \to D(Y)$ gives an equivalence. Then there exists an object $\mathcal{P} \in D(X \times Y)$ such that Φ is isomorphic to the functor $\Phi^{\mathcal{P}}_{X \to Y}$. Moreover $\mathcal{P}$ is uniquely determined up to isomorphism.*

$\mathcal{P}$ is called a kernel of Φ. The problem is classifying the Fourier-Mukai partners of X. The followings are known results.

- $\dim X = 1$

  In this case, it is easy to show $FM(X) = \{X\}$.

- $\dim X = 2$

  In this case, $FM(X) = \{X\}$ except X is a $K3$ surface or an Abelian surface, or an elliptic surface. When X is one of such varieties, $FM(X)$ are given by some moduli spaces of stable sheaves. These results are shown by Bridgeland-Maciocia [4] and Kawamata [7].

- X is a general type or $\pm K_X$ is ample

  When X is a general type, Kawamata [7] showed $Y \in FM(X)$ if and only if X and Y are K-equivalent. When $\pm K_X$ is ample, then $FM(X) = \{X\}$. This is a result of Bondal-Orlov [1].

In these results, we can see the common methods in treating this problem, summarized as follows:

- If K_X has much information, then we can reconstruct (general) closed points $\{\mathcal{O}_x\}_{x \in X}$
- If K_X has no information, then we can use Torelli theorem

The main purpose of this article is to generalize these methods, and the main idea is the following:

"If there exists $E \in |mK_X|$, then use E to reduce the problem to lower dimensional case."

2 Correspondence of canonical divisors

In this section we compare the canonical divisors of X and Y, when $Y \in FM(X)$, and state the main theorem. Firstly we define the Serre functor.

Definition 2.1 *We define S_X as*

$$S_X := \otimes\omega_X[\dim X]\colon D(X) \to D(X).$$

S_X satisfies the following categorical property

$$\mathrm{Hom}(E, F) \cong \mathrm{Hom}(F, S_X(E)),$$

and characterized by this property. Therefore if $\Phi\colon D(X) \to D(Y)$ gives an equivalence, then we have an isomorphism of functors

$$\Phi \circ S_X \cong S_Y \circ \Phi.$$

Since the kernel of the left hand side is given by $\mathcal{P} \otimes f^*\omega_X[\dim X]$ and right hand side is given by $\mathcal{P} \otimes g^*\omega_Y[\dim Y]$, we have an isomorphism by Orlov's theorem

$$\mathcal{P} \otimes f^*\omega_X[\dim X] \xrightarrow{\cong} \mathcal{P} \otimes g^*\omega_Y[\dim Y].$$

So it follows that $\dim X = \dim Y$ and for all $m \in \mathbb{Z}$ we have an isomorphism

$$\rho_m\colon \mathcal{P} \otimes \mathcal{O}(mf^*K_X) \xrightarrow{\cong} \mathcal{P} \otimes \mathcal{O}(mg^*K_Y).$$

On the other hand, since S_X is a categorical invariant, we have the isomorphism of natural transforms

$$\tau_m\colon \mathrm{Nat}(\mathrm{id}_X, S_X^m[-dm]) \xrightarrow{\cong} \mathrm{Nat}(\mathrm{id}_Y, S_Y^m[-dm]).$$

Here $d = \dim X$ and Nat means natural transforms. Note that $\mathrm{Nat}(\mathrm{id}_X, S_X^m[-dm])$ contains $H^0(X, mK_X)$ as a linear subspace and it is easy to show that the above isomorphism preserve these subspaces. So we have an isomorphism

$$H^0(X, mK_X) \xrightarrow{\cong} H^0(Y, mK_Y).$$

Take $\sigma \in H^0(X, mK_X)$ and $\mathrm{div}(\sigma) = E \in |mK_X|$. Let $\sigma^\dagger \in H^0(Y, mK_Y)$ corresponds to σ and $\mathrm{div}(\sigma^\dagger) = E^\dagger \in |mK_Y|$. We want to compare E and $E^\dagger$. It is easy to show that Φ preserves these supports. For $Z \hookrightarrow X$ closed subscheme, we define $D_Z(X)$ as follows:

$$D_Z(X) := \{a \in D(X) \mid \mathrm{Supp}\, a := \cup \mathrm{Supp}\, H^i(a) \subset Z\}.$$

We have the following lemma.

Lemma 2.2 *Φ takes $D_E(X)$ to $D_{E^\dagger}(Y)$.*

(*Proof*)Take $a \in \mathrm{Coh}(X) \cap D_E(X)$. Then

$$\sigma^N(a)\colon a \to a \otimes \mathcal{O}(NmK_X)$$

are zero-maps for sufficiently large N. Then

$$(\sigma^\dagger)^N(\Phi(a))\colon \Phi(a) \to \Phi(a) \otimes \mathcal{O}(NmK_Y)$$

are also zero-maps. This implies $\mathrm{Supp}\,\Phi(a) \subset E^\dagger$, and since $D_E(X)$ is generated by $\mathrm{Coh}(X) \cap D_E(X)$, the lemma follows. □

Unfortunately $D(E)$ is far from $D_E(X)$ and we want to compare $D(E)$ and $D(E^\dagger)$. If $D(E)$ and $D(E^\dagger)$ are equivalent, then the relation between E and $E^\dagger$ gives some information of the relation between X and Y. For the sake of applications, it is convenient to formulate the theorem on the complete intersections of these divisors. Take $E_i \in |m_iK_X|$ for $i = 1, 2 \cdots, n$. Here n is an arbitrary natural number. Take a connected component $C \subset \bigcap_{i=1}^n E_i$. Then by the same argument of the lemma, there exists an unique connected component $C^\dagger \subset \bigcap_{i=1}^n E_i^\dagger$ such that Φ takes $D_C(X)$ to $D_{C^\dagger}(Y)$. We assume the following conditions:

- C and $C^\dagger$ are complete intersections.
- $\mathcal{T}or_i^{\mathcal{O}_{X\times Y}}(H^k(\mathcal{P}), \mathcal{O}_{C\times C^\dagger}) = \mathcal{T}or_i^{\mathcal{O}_{X\times Y}}(H^k(\mathcal{E}), \mathcal{O}_{C\times C^\dagger}) = 0$ for all k and $i > 0$.

Here $\mathcal{E}$ is a kernel ofΦ^{-1}. These conditions are satisfied, for example, $|m_iK_X|$ are free and E_i are generic members. Main theorem is the following:

Theorem 2.3 *Under the above conditions, there exists an equivalence* $\Phi_C\colon D(C) \to D(C^\dagger)$ *such that the following diagram is commutative:*

$$\begin{array}{ccccc}
D(X) & \xrightarrow{\mathbf{L}i_C^*} & D(C) & \xrightarrow{i_{C*}} & D(X) \\
{\scriptstyle\Phi}\downarrow & & {\scriptstyle\Phi_C}\downarrow & & {\scriptstyle\Phi}\downarrow \\
D(Y) & \xrightarrow{\mathbf{L}i_{C^\dagger}^*} & D(C^\dagger) & \xrightarrow{i_{C^\dagger *}} & D(Y)
\end{array}$$

Here i_C, $i_{C^\dagger}$ *are inclusions.*

3 Outline of the proof of the main theorem

Take $\sigma \in H^0(X, K_X)$, $E = \mathrm{div}(\sigma)$ as in the previous section. Let $\mathcal{P} \in D(X \times Y)$ be a kernel of $\Phi\colon D(X) \to D(Y)$.

Step 1 *The following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{P}\otimes\mathcal{O}(-mf^*K_X) & \xrightarrow{id\otimes\sigma} & \mathcal{P} \\
{\scriptstyle\rho_{-m}}\downarrow & & \| \\
\mathcal{P}\otimes\mathcal{O}(-mg^*K_Y) & \xrightarrow{id\otimes\sigma^\dagger} & \mathcal{P}.
\end{array}$$

(*Idea of the proof*) By definition induced diagram of natural transforms is commutative, i.e. $\mathrm{id} \circ \sigma = \sigma^{\dagger} \circ \mathrm{id} \circ \tau_{-m}$. We can describe $\mathcal{P}$ in terms of Φ by using the proof of the Orlov's theorem. Combine these results. □

By taking cones, Step 1 implies there exists an isomorphism

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{E \times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X \times E^{\dagger}}.$$

Take σ_i, E_i, and C, $C^{\dagger}$ as in the previous section. Then we have an isomorphism

$$\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y} \cong \mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{X \times C^{\dagger}} \quad \cdots (\star).$$

Here we used the assumption C and $C^{\dagger}$ are complete intersections. Now we have

Step 2 *There exists some object $\mathcal{P}_C \in D(C \times C^{\dagger})$ such that*

$$\mathcal{P} \overset{L}{\otimes} \mathcal{O}_{C \times Y} \cong \mathcal{P} \overset{L}{\otimes} \mathcal{O}_{X \times C^{\dagger}} \cong i_{C \times C^{\dagger} *} \mathcal{P}_C.$$

(*Idea of the proof*) By applying $\overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y}$ to $(\star)$, we can see $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y}$ is a direct summand of $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times C^{\dagger}}$, which is a push-forward from $C \times C^{\dagger}$. Now we use the assumption of the higher $\mathcal{T}or$ to show $\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathcal{O}_{C \times Y}$ is actually push-foward from $C \times C^{\dagger}$. □

Step 3 *Let $\Phi_C := \Phi^{\mathcal{P}_C}_{C \to C^{\dagger}} \colon D(C) \to D(C^{\dagger})$. Then Φ_C gives a desired equivalence.*

(*Proof*) Firstly we show the commutativity of the diagram. This easily follows from the isomorphism of Step 2. Secondly we show that Φ_C gives an equivalence. Let $\Psi_{C^{\dagger}} \colon D(C^{\dagger}) \to D(C)$ be a functor defined from $\Psi := \Phi^{-1}$ as in the same way. Then the commutativity of the diagram implies

$$\Psi_{C^{\dagger}} \circ \Phi_C(\mathcal{O}_x) = \mathcal{O}_x, \quad \Psi_{C^{\dagger}} \circ \Phi_C(\mathcal{O}_C) = \mathcal{O}_C.$$

These imply $\Psi_{C^{\dagger}} \circ \Phi_C = \mathrm{id}$, since the kernel of $\Psi_{C^{\dagger}} \circ \Phi_C$ is a diagonal. Similarly $\Phi_C \circ \Psi_{C^{\dagger}} = \mathrm{id}$, and the proof is completed. □

4 Applications

We can apply the Main theorem to the classification of $FM(X)$. Note that the minimal 3-fold X has an algebraic fiber space structure

$$\pi \colon X \to Z := \operatorname{Proj} \oplus_{m \geq 0} H^0(X, mK_X).$$

π is called Iitaka fibration. Let $X_{\bar{\eta}}$ be geometric generic fiber of π. In this section, we assume the following:

- X is a smooth minimal 3-fold of $\kappa(X) = 1$.
- $X_{\bar{\eta}}$ is a $K3$ surface or an Abelian surface.
- All the fibers of π is irreducible and reduced.

Before we state the theorem, we give a definition

Definition 4.1 *Let $H \in \mathrm{Pic}(X)$ be a polarization. We denote $M^H(X/Z)$ by relative moduli space of stable sheaves with respect to H. An irreducible component $M \subset M^H(X/Z)$ is fine if $M \to Z$ is projective and there exists an universal family on $X \times_Z M$.*

The main theorem of this section is the following:

Theorem 4.2 *In the above situation, $Y \in FM(X)$ if and only if there exists some polarization H on X, and an irreducible component $M \subset M^H(X/Z)$ which is fine and relative dimension =2, such that Y and M are connected by finite number of flops.*

(*Out line of the proof*) Let $\Phi\colon D(X) \to D(Y)$ be as in the previous sections. The isomorphism $H^0(X, mK_X) \cong H^0(Y, mK_Y)$ preserves graded ring structures, so we have

$$Z := \mathrm{Proj} \oplus_{m\geq 0} H^0(X, mK_X) \cong \mathrm{Proj} \oplus_{m\geq 0} H^0(Y, mK_Y).$$

Let $\pi_X\colon X \to Z$, $\pi_Y\colon Y \to Z$ be Iitaka fibrations. Then the main theorem implies, for general points $p \in Z$, X_p, Y_p fibers at p, there exists an equivalence $\Phi_p\colon D(X_p) \to D(Y_p)$ such that the diagram

$$\begin{array}{ccccc} D(X) & \xrightarrow{\mathbf{L}i^*_{X_p}} & D(X_p) & \xrightarrow{i_{X_p*}} & D(X) \\ \Phi\big\downarrow & & \Phi_p\big\downarrow & & \Phi\big\downarrow \\ D(Y) & \xrightarrow{\mathbf{L}i^*_{Y_p}} & D(Y_p) & \xrightarrow{i_{Y_p*}} & D(Y). \end{array}$$

commutes. For the sake of simplicity, we assume X_p is a $K3$ surface. Now we use the general facts of derived categories and singular cohomologies. For a functor $\Phi^{\mathcal{P}}_{X\to Y}\colon D(X) \to D(Y)$, which is not necessary equivalent, we can define a linear map

$$\phi^{\mathcal{P}}_{X\to Y}\colon H^*(X, \mathbb{Q}) := \oplus_{k\geq 0} H^k(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}).$$

$\phi^{\mathcal{P}}_{X\to Y}$ is defined by the algebraic cycle $f^*\sqrt{\mathrm{td}_X}\,\mathrm{ch}(\mathcal{P}) g^*\sqrt{\mathrm{td}_Y} \in H^*(X \times Y, \mathbb{Q})$, and the following diagram commutes:

$$\begin{array}{ccc} D(X) & \xrightarrow{\Phi} & D(Y) \\ {\scriptstyle \mathrm{ch}(*)\sqrt{\mathrm{td}_X}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ch}(*)\sqrt{\mathrm{td}_Y}} \\ H^*(X, \mathbb{Q}) & \xrightarrow{\phi} & H^*(Y, \mathbb{Q}). \end{array}$$

Moreover the correspondence $\Phi^{\mathcal{P}}_{X\to Y} \to \phi^{\mathcal{P}}_{X\to Y}$ is functorial. Applying these facts to our situations, we have the commutative diagram

$$\begin{array}{ccccc} H^*(X,\mathbb{Q}) & \xrightarrow{i^*_{X_p}} & H^*(X_p,\mathbb{Q}) & \xrightarrow{i_{X_p*}} & H^*(X,\mathbb{Q}) \\ \phi\downarrow & & \phi_p\downarrow & & \phi\downarrow \\ H^*(Y,\mathbb{Q}) & \xrightarrow{i^*_{Y_p}} & H^*(Y_p,\mathbb{Q}) & \xrightarrow{i_{Y_p*}} & H^*(Y,\mathbb{Q}). \end{array}$$

As in [9], ϕ_p is defined over $\mathbb{Z}$ and preserves inner products. Here inner product on $H^*(X_p,\mathbb{Z}) = H^0 \oplus H^2 \oplus H^4$ is given by $(r,l,s)\cdot(r',l',s') = ll' - rs' - r's$. Now take $(0,0,1) \in H^*(Y_p,\mathbb{Z})$ and let $(r_p,l_p,s_p) := \phi_p^{-1}(0,0,1)$. Then by composing suitable equivalence if necessary, we may assume $r_p \geq 2$, l_p is ample. Moreover we can show there exists a polarization H on X such that $H|_{X_p} = dl_p$ for some $d > 0$. Let $M \subset M^H(X/Z)$ be an irreducible component which contains stable sheaves on X_p whose Mukai vector $:= \sqrt{\operatorname{td} X_p}\operatorname{ch}(*)$ equals to (r_p,l_p,s_p). Then as in [9], M is non-empty, and we can check the condition of the existence of the universal sheaves in [9]. Moreover, since all the fibers of π_X are irreducible and reduced, M is projective by the argument of [6, Remark 4.6.8]. Therefore we can conclude M is fine. By [5], M is smooth and the universal sheaf gives an equivalence $D(M) \to D(X)$. By composing this equivalence with Φ, we can reduce the problem to the following:

"If $(r_p,l_p,s_p) = (0,0,1)$, then X and Y are connected by finite number of flops."

Since X and Y are minimal 3-fold, it is enough to show X and Y are birational. Note that $(0,0,1)^{\perp}/(0,0,1) \cong H^2(X_p,\mathbb{Z})$, so there exists a Zariski open set $Z^0 \subset Z$ such that we have a Hodge isometry

$$\tilde{\phi}\colon R^2\pi_{X*}\mathbb{Z}_X|_{Z^0} \to R^2\pi_{Y*}\mathbb{Z}_Y|_{Z^0}.$$

Now using the results of the families of $K3$ surfaces in [10], we can show that, by shrinking Z^0 if necessary, there exists a Hodge isometry $\tilde{\phi}'\colon R^2\pi_{Y*}\mathbb{Z}_Y|_{Z^0} \to R^2\pi_{Y*}\mathbb{Z}_Y|_{Z^0}$ such that the composition $\tilde{\phi}' \circ \tilde{\phi}$ is an effective Hodge isometry. By Torelli theorem of $K3$ surfaces, there exists an isomorphism $f_p\colon Y_p \to X_p$ such that $(\tilde{\phi}' \circ \tilde{\phi})_p = f_p^*$. Then $\{f_p\}_{p\in Z^0}$ gives a section of $\operatorname{Isom}_{Z^0}(Y,X) \to Z^0$. □

5 Appendix

In the case of $\kappa(X) = 2$, we have the following result.

Theorem 5.1 *Let X be a smooth projective 3-fold of $\kappa(X) = 2$. Then $Y \in FM(X)$ if and only if one of the following holds:*

(i)X and Y are connected by finite number of flops.

(ii)There exists a following diagram:

$$\begin{array}{ccccc} Y \overset{\text{flops}}{\dashrightarrow} J^H(d) & & & & M \overset{\text{flops}}{\dashrightarrow} X \\ & \searrow^{\pi'} & & \swarrow_{\pi} & \\ & & S & & \end{array}$$

where $\pi\colon X^+ \to S$ is an elliptic fibration with $\omega_M \equiv_\pi 0$, $H \in \operatorname{Pic}(M)$ is a polariza-

tion, $d \in \mathbb{Z}$, and $J^H(d) \subset M^H(M/S)$ is an irreducible component which is fine and contains line bundles of degree d on smooth fibers of π.

In the case of $\kappa(X) = 1$, If we eliminate the assumption "all the fibers of the Iitaka fibration are irreducible and reduced" in the previous section, we have the following result.

Theorem 5.2 *Assume X is a minimal 3-fold of $\kappa(X) = 1$, and generic fiber of its Iitaka fibration is a $K3$ surface. Then $Y \in FM(X)$ if and only if of the following holds.*

(i)There exists a polarization H on X and an irreducible component $M \subset M^H(X/Z)$, which is fine and relative dimension two, such that Y and M are connected by finite number of flops.

(ii)There exists a polarization H on Y and an irreducible component $M \subset M^H(Y/Z)$, which is fine and relative dimension two, such that X and M are connected by finite number of flops.

I hope that $Y \in FM(X)$ if and only if (i) holds, but unfortunately I couldn't prove. The problem is whether we can take moduli space which is projective.

By the same method, we can study $FM(X)$ when $X_{\bar{\eta}}$ is an Enriques surface or a bielliptic surface. Let X be a good minimal model (i.e.K_X is semi-ample) and $\pi_X \colon X \to Z$ be its Iitaka fibration. Let $m := \min\{i \mid \omega_{X_{\bar{\eta}}}^{\otimes i} \cong \mathcal{O}_{X_{\bar{\eta}}}\}$. Then there exists a Zariski open subset $Z^0 \subset Z$ such that $\omega_{X^0}^{\otimes m} \cong \mathcal{O}_{X^0}$, where $X^0 = \pi_X^{-1}(Z^0)$. Let

$$p_X \colon \widetilde{X}^0 := \mathbf{Spec}_{\mathcal{O}_{X^0}}\left(\bigoplus_{i=0}^{m-1} \omega_{X^0}^{\otimes(-i)}\right) \to X^0$$

be its canonical cover. Let $Y \in FM(X)$ and $\pi_Y \colon Y \to Z$ be its Iitaka fibration. Then, $\min\{i \mid \omega_{Y_{\bar{\eta}}}^{\otimes i} \cong \mathcal{O}_{Y_{\bar{\eta}}}\}$ is also m because general fiber of π_Y is also a Fourier-Mukai partner of general fiber of π_X. Let us take a canonical cover $\pi_Y \colon \widetilde{Y}^0 \to Y^0$. Then the equivalence $\Phi \colon D(X) \to D(Y)$ gives an equivalence $\Phi^0 \colon D(X^0) \to D(Y^0)$. Let $G = \mathrm{Gal}(\widetilde{X}^0/X^0) \cong \mathrm{Gal}(\widetilde{Y}^0/Y^0) \cong \mathbb{Z}/m\mathbb{Z}$. Let $p_{X,p} := p_X|_{\widetilde{X}_p}$, $p_{Y,p} := p_Y|_{\widetilde{Y}_p}$.

Definition 5.3 *A functor $\widetilde{\Phi}^0 \colon D(\widetilde{X}^0) \to D(\widetilde{Y}^0)$ is G-equivariant if there exists some group isomorphism $\sigma \colon G \to G$ such that the following diagram commutes for all $g \in G$:*

$$\begin{array}{ccc} D(\widetilde{X}^0) & \xrightarrow{\widetilde{\Phi}^0} & D(\widetilde{Y}^0) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle \sigma(g)^*} \\ D(\widetilde{X}^0) & \xrightarrow{\widetilde{\Phi}^0} & D(\widetilde{Y}^0), \end{array}$$

By combining the method of [3] and our method, we can easily show the following theorem.

Theorem 5.4 (*By shrinking Z^0 if necessary,*) *There exists a G-equivariant equivalence $\widetilde{\Phi}^0 \colon D(\widetilde{X}^0) \to D(\widetilde{Y}^0)$ such that the following diagram is commutative:*

$$(\diamondsuit) \qquad \begin{tikzcd} D(X^0) \arrow[r, "p_X^*"] \arrow[d, "\Phi^0"'] & D(\widetilde{X}^0) \arrow[r, "p_{X*}"] \arrow[d, "\widetilde{\Phi}^0"'] & D(X^0) \arrow[d, "\Phi^0"'] \\ D(Y^0) \arrow[r, "p_Y^*"] & D(\widetilde{Y}^0) \arrow[r, "p_{Y*}"] & D(Y^0). \end{tikzcd}$$

Moreover there exists a G-equivariant equivalence $\widetilde{\Phi}_p \colon D(\widetilde{X}_p) \to D(\widetilde{Y}_p)$ such that the following diagrams commute:

$$(\diamondsuit') \qquad \begin{tikzcd} D(\widetilde{X}^0) \arrow[r, "\mathbf{L}i^*_{\widetilde{X}_p}"] \arrow[d, "\widetilde{\Phi}^0"'] & D(\widetilde{X}_p) \arrow[r, "i_{\widetilde{X}_p *}"] \arrow[d, "\widetilde{\Phi}_p"'] & D(\widetilde{X}^0) \arrow[d, "\widetilde{\Phi}^0"'] \\ D(\widetilde{Y}^0) \arrow[r, "\mathbf{L}i^*_{\widetilde{Y}_p}"] & D(\widetilde{Y}_p) \arrow[r, "i_{\widetilde{Y}_p *}"] & D(\widetilde{Y}^0), \end{tikzcd}$$

$$(\diamondsuit'') \qquad \begin{tikzcd} D(X_p) \arrow[r, "p^*_{X,p}"] \arrow[d, "\Phi_p"'] & D(\widetilde{X}_p) \arrow[r, "p_{X,p*}"] \arrow[d, "\widetilde{\Phi}_p"'] & D(X_p) \arrow[d, "\Phi_p"'] \\ D(Y_p) \arrow[r, "p^*_{Y,p}"] & D(\widetilde{Y}_p) \arrow[r, "p_{Y,p*}"] & D(Y_p). \end{tikzcd}$$

Assume the following:

- X is a smooth minimal 3-fold of $\kappa(X) = 1$.
- $X_{\bar\eta}$ is an Enriques surface or a bielliptic surface.
- If $X_{\bar\eta}$ is a bielliptic surface, all the fibers of π_X are irreducible and reduced.

Under these conditions, we can study $FM(X)$ by using the above theorem. In fact we have the following theorem:

Theorem 5.5 *Under the above conditions $Y \in FM(X)$ if and only if there exists a polarization H on X and an irreducible component $M \subset M^H(X/Z)$ which satisfies*

- *M is fine and $M \to Z$ is relative dimension two.*
- *For all $x \in M$, corresponding stable sheaf E_x satisfies $E_x \otimes \omega_X \cong E_x$.*

such that Y and M are connected by finite number of flops.

Problem 5.6 By the classification of $FM(X)$ in the surface case, $FM(X_p) = \{X_p\}$ in this case. Are there any member in $FM(X)$ which is not birational to X?

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